Introduction to Probability and Statistics

Topic (4): " Probability and Probability distribution"

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1 Events and the Sample Space

Main concepts:

1) Random experiment: if the outcome of it can't determine certainly before the performance of experiment.

Example (1): Tossing a coin, Rolling a dice.

2) Sample space (S): the collection of the set of all possible outcome of random experiment.

Example (2): a) What is the sample space for tossing a coin?

b) What is the sample space for rolling a dice?

Example (3): Toss a coin then roll a dice, write the sample space (S)?

Example (4): Toss two coins, write the sample space?

Example (5): Toss three coins, write the sample space?

Example (6): Toss a coin until you get a head, write the sample space?

Example (7): Roll two dice and write the sample space?

Exercise (1): Find the sample space for the gender of the children if a family has three children. Use B for boy and G for girl.

3) Event is a subset from the sample space.

Example (8): Roll a dice.

a) Write the event B, where B is the odd number.

b) Write the event C, where C is the even number.

c) Write the event D, where D is numbers less than 5.

Example (9): Roll two dices.

a) Write the event A, where A is the sum of two numbers is equal to 4.

b) Write the event B, where B is the sum of two numbers which is less than or equal 5.

c) Write the event C, where C is the sum of two numbers which is greater than or equal 10.

Types of events:

1) Simple event: consist one element. Example: {4}.

2) Composite event: consist more than one element. Example: {4, 5}.

Example (10): Rolling two die the event A = {(3, 5)} is a ----------------.

Exercise (2): Consider the experiment of choosing at random a digit from the digits $0, 1, 2, \cdots, 9$.

1. What is the sample space of this experiment?
2. Find the elements of each of the following events:

(a) A = the number chosen is less than or equal to 3.

(b) B = the number chosen is between 4 and 6, inclusive.

(c) C = the number chosen is greater than or equal to 7.

(d) D = the number chosen is less than 4 or larger than 7.

(e) E = the number chosen is an even number.

2 Calculating Probability using Simple Events

$P(A)$: Probability of an event A.

$$P(A) = \frac{\text{number of el}ements\ in\ A}{\text{number of el}ements\ in\ S}$$

Note: The probability of any event $A \subseteq S$ must satisfy:

a. $0 \leq P(A) \leq 1$
b. $P(S) = 1$, Sum of probability of all simple event in S is equal to 1.
c. $P(\emptyset) = 0$

Example (11): Roll a dice. Find the probability that we get odd number.

Example (12): We have 4 events E_1, E_2, E_3, E_4. $P(E_1) = 0.1, P(E_2) = 0.3, P(E_3) = 0.2$. Find $P(E_4)$.

Example (13): Toss two coins.

1) Find the probability of observing exactly one head.

2) Find the probability of observing at least one head.

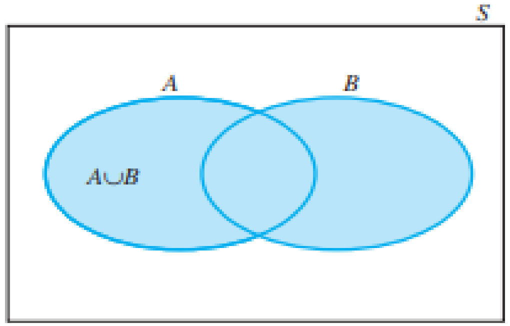
3) Find the probability of observing at most one head.

3 Event Relation and Probability Rules

Calculating Probabilities for Unions and Complements

Let A and B be two events defined on the sample space S.

Union of Two Events: $(A \cup B)$ or $(A + B)$
The event $A \cup B$ consists of all outcomes in A **or** in B **or** in both A and B. The event $A \cup B$ occurs if A occurs, **or** B occurs, **or** both A and B occur.

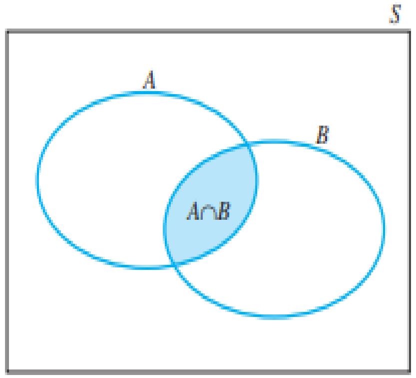


Venn diagram of $A \cup B$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Intersection of Two Events: $(A \cap B)$

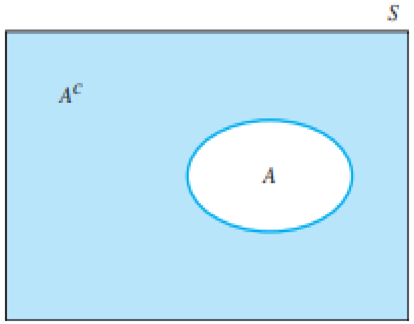
The event $A \cap B$ consists of all outcomes in A **and** B. The event $A \cap B$ occurs if both A **and** B occur.



Venn diagram of $A \cap B$

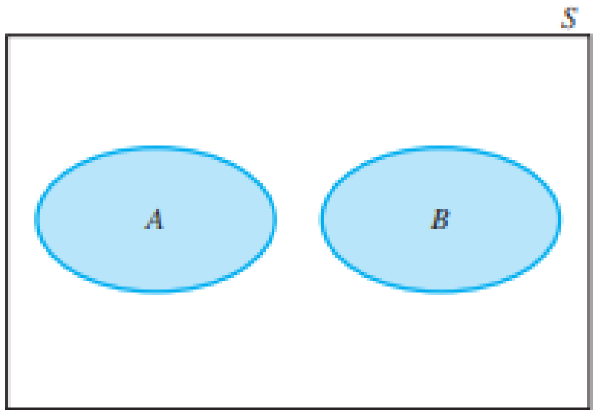
Complement of an Event: A^c or A'

The complement of the event A is denoted by A^c. The event A^c consists of all outcomes of S but are not in A. The event A^c occurs if A does not occur.



$$P(A^c) = 1 - P(A)$$

Definition: Two events A and B are mutually exclusive (disjoint). If $A \cap B = \emptyset \equiv \{\ \}$.



$$A \cap B = \emptyset$$

$$P(A \cap B) = 0$$

Note: $P(\emptyset)$ = 0

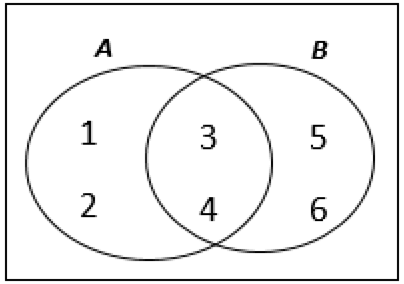
Note: when two events A and B are mutually exclusive, then $P(A \cap B) = 0$ and the addition rule simplifies to $P(A \cup B) = P(A) + P(B)$

Example (14): Roll a dice.

A: is odd and B: is even. Are A and B disjoint?

Example (15): Let C = {1, 2, 3} and D = {2, 4, 5}. Are C and D disjoint?

Example (16): A dice.



Find:

1) $A \cup B$
2) $A \cap B$
3) A^c

Example (17): we have blood group (A, B, AB, O) for patients, $P(A) = 0.41, P(B) = 0.1, P(AB) = 0.04, P(O) = 0.45$. We select a patient randomly. Find the probability that his blood group will be either type A or B.

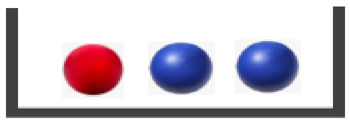
4 Independence, Conditional Probability, and the Multiplication Rule

Definition: Two events A and B are independent if $P(A \cap B) = P(A)P(B)$.

Note: when two events A and B are independent, then

$$P(A \cap B) = P(A)P(B) \text{ and then } P(A \cup B) = P(A) + P(B) - P(A)P(B).$$

Example (18): A box contains 3 balls 1 red and 2 blue.



We select two balls one at time. Find the probability that two balls are blue.

If the balls were selected without replacement:

If the balls were selected with replacement:

Example (19): Rolling a dice. Find the events and their probability.

A: Observe number 2.

B: Even numbers

C: Observe a number greater than 2.

D: Observe both A and B.

E: Observe A or B or both.

F: Observe both A and C.

Example (20): A sample space S consists of five simple events with these probabilities:

$$P(E_1) = P(E_2) = 0.15, P(E_3) = 0.4, P(E_4) = 2P(E_5).$$

1) Find $P(E_4)$ and $P(E_5)$.

2) let A = $\{E_1, E_2, E_4\}$, B = $\{E_2, E_3\}$ and C = $\{E_1, E_2\}$. Find the probability of A, B, C.

3) Find probability that both event A and B occur.

4) Find probability that both event A or B occur.

Exercise (3): A sample space contains 10 simple events: $E_1, E_2, \ldots, E_{10}$. If $P(E_1) = 3P(E_2) = 0.45$ and the remaining simple events are equiprobable, find the probabilities of these remaining simple events.

Notes: Let A and B be two events

- Probability both occur: $P(A \cap B)$
- Probability at least one event occur: $P(A \cup B)$
- Probability of A but not B occur: $P(A \cap B^c)$
- Probability of neither A or B occur: $P(A^c \cap B^c)$
- Probability Exactly one event occur: $P(A \cap B^c) \cup P(A^c \cap B)$

Example (21): Roll a dice.

$A = \{1, 2, 3\}$ and $B = \{3, 4, 6\}$.

Find: $A \cap B$, $A \cup B$, A^c, probability of event A and B, probability of event A or B, probability of event not B.

Example (22): Toss two coins.

A: At least one head.

B: At least one tail.

C: Three head.

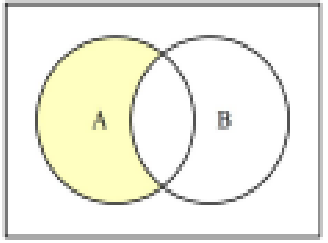
Find: $P(A \cap B), P(A \cup B), P(A^C), P(C)$.

Probability rules:

The difference of two events A and B, denoted by $A - B$, is the set of all elementary outcomes that are in A but not in B.

1) $P(A - B) = P(A \cap B^c) = P(A) - P(A \cap B)$

2) $P(A^c \cap B) = P(B) - P(A \cap B)$

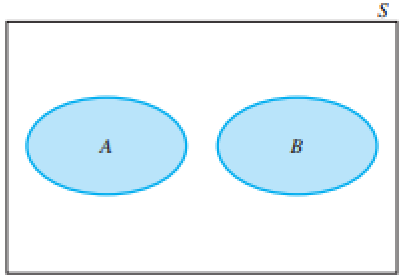


Note: If A and B are disjoint

1) $P(A \cap B) = 0$

2) $P(A \cup B) = P(A) + P(B)$

Two disjoint events



Demorgan laws:

1) $P(A \cup B)^c = P(A^c \cap B^c)$

2) $P(A \cap B)^c = P(A^c \cup B^c)$

Example (23): Let A and B be two events, such that:

$P(A) = 0.5, P(B) = 0.2, P(A \cap B) = 0.1$

Find:

1) $P(A \cup B) =$

2) $P(B^c) =$

3) $P(A \cap B^c) =$

4) $P(A^c \cap B^c) =$

5) $P(A^c \cup B^c) =$

6) $P(A \cup B^c) =$

Conditional probability:

Definition: the conditional probability that an event A given that event B has occurred.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Example (24): Roll a dice.

1) Find the probability that we get number 3.

2) Given that the outcome is odd. Find the probability that we get number 3.

Example (25): Let A and B be two events such that:

$$P(A) = 0.55, P(B) = 0.3, P(A \cap B) = 0.25$$

Find:

1) $P(A|B) =$

2) $P(B|A) =$

3) $P(A|B^c) =$

4) $P(A^c|B^c) =$

Example (26): Suppose we have 20 bulbs as follows:

	Good	Defective
Red	5	8
Yellow	3	4

We select bulbs randomly

1) Find the probability we get red bulbs.

2) Find the probability we get good bulbs.

3) Find the probability we get good and red bulbs.

4) Find the probability we get good or red bulbs.

5) Given that the bulb is good. Find the probability it was red bulbs.

6) Find the probability that we get good bulbs. If it was red.

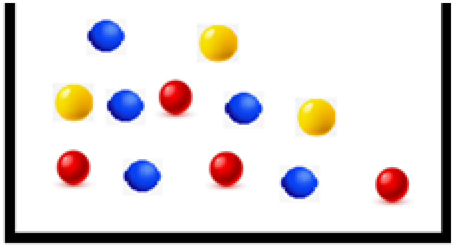
Exercise (4): The following table gives the two-way classification of 500 students based on gender and whether or not they suffer from math anxiety (تعاني من قلق في الرياضيات).

Gender	Yes	No
Male	145	95
Female	190	70

If a student randomly selected.

1. What is the probability that this student is female given that she does not suffer from math anxiety?
2. What is the probability that this student suffers from math anxiety given that he is a male?
3. What is the probability that this student is a male?
4. What is the probability that this student does not suffer from math anxiety?

Example (27): A box contains 4 red balls, 3 yellow balls, 5 blue balls. We withdraw two balls randomly one at a time without replacement.



1) Find the probability that the first ball is red and the second is blue.

2) Find the probability that the two balls are red.

3) Find the probability that one ball is red and one is blue.

4) Find the probability that the second drawn ball is red.

5) Find the probability that both balls are the same color.

Note: If A and B are independent events, then $P(A|B) = \mathrm{P}(A)$ and $P(B|A) = \mathrm{P}(B)$

Example (28): Toss two coins.

A: Head on first coin.

B: Tail on Second coin.

Are A and B independent?

Example (29): Let A and B be two independent event, such that: $P(A)$ = 0.5 and $P(B)$ = 0.4. Find:

a) $P(A \cup B) =$

b) $P(A^c \cap B) =$

c) $P(A|B) =$

d) $P(A^c|B) =$

Example (30): Let A and B be two events such that: $P(A)$ = 0.5, $P(B)$ = 0.6 and $P(A \cup B)$ = 0.7 are A and B independent? Why?

Exercise (5): 4.42 An experiment can result in one of five equally likely simple events, $E_1, E_2, \ldots, E_5$. Events A, B, and C are defined as follows:

A: E_1, E_3.

B: E_1, E_2, E_4, E_5.

C: E_3, E_4.

Find the probabilities associated with these compound events by listing the simple events in each.

a) A^c

b) $A \cap B$

c) $B \cap C$

d) $A \cup B$

e) $B|C$

f) $A|B$

g) $A \cup B \cup C$

h) $(A \cap B)^c$

Exercise (6): Suppose that $P(A)$ = 0.4 and $P(B)$ = 0.2. If events A and B are independent, find these probabilities:
a. $P(A \cap B)$ **b.** $P(A \cup B)$

Exercise (7): Suppose that $P(A) = 0.3$ and $P(B) = 0.5$. If events A and B are mutually exclusive, find these probabilities:
a. $P(A \cap B)$ **b.** $P(A \cup B)$

Exercise (8): Suppose that $P(A) = 0.4$ and $P(A \cap B) = 0.12$.
a. Find $P(B|A)$.
b. Are events A and B mutually exclusive?
c. If $P(B)$ = 0.3, are events A and B independent?

Exercise (9): An experiment can result in one or both of events A and B with the probabilities shown in this probability table:

	A	A^c
B	0.34	0.46
B^c	0.15	0.05

1) Find the following probabilities:

a. $P(A)$ **b.** $P(B)$ **c.** $P(A \cap B)$ **d.** $P(A \cup B)$ **e.** $P(A|B)$ **f.** $P(B|A)$

2) Are A and B independent?

5 Discrete Random Variables and their Probability distributions

Definition: Random variable ($r.v$): given a random experiment with a sample space (S), the function x which assigns values (numbers) to each event (outcomes) in S.

Example (31): Toss two coins. The random experiment defined x to be number of head.

Example (32): Rolling two dice. x: sum of numbers.

Example (33): Toss three coins. The random experiment defined x to be number of head.

Probability distribution: Table summarized the values of x and their probability.

Definition: $P(X = x)$ is called probability distribution function.

Example (34): Toss two coins, define x to be number of head.

1) Find the probability distribution.

2) Find $P(X \leq 1) =$

Properties of $P(x)$:

1) $P(x) > 0, \ x \in R$
$P(x) = 0, \ x \notin R$
2) $\sum P(x) = 1, \ \forall x \in R$

Mean and variance of $r.v \ x$:

Mean = μ = $E(x)$ → Expected value of x.

$E(x) = \sum[x\,P(x)], \ x \in R$

Variance = σ^2 = $E(x-\mu)^2 = \sum[(x-\mu)^2 P(x)], \ x \in R$

$$\sigma^2 = Ex^2 - \mu^2, \qquad \text{where } Ex^2 = \sum[x^2\,P(x)]; \ x \in R$$

Standard deviation $\sigma = \sqrt{\sigma^2}$

Example (35): Let x be a random variable with probability distribution:

x	**0**	**1**	**2**	**o.w**
$P(X = x)$	1/4	2/4	1/4	0

1) Find the mean of x.

2) Find the variance of x.

3) Find the standard deviation of x.

Example (36): Let x be a random variable with mean 3 and standard deviation 5. Find Ex^2.

Example (37): Let x be a $r.v$ with probability distribution:

x	**0**	**1**	**2**	**3**	**4**	**5**
$P(X = x)$	0.34	0.3	0.2	c	0.05	0.01

1) Find c.

2) $P(X = 1)$ =

3) $P(X \geq 3) =$

4) $P(X > 3) =$

5) $P(X > 1) =$

6) $P(X \geq 1) =$

7) $P(0 < X \leq 3) =$

8) $P(0 \leq X < 2) =$

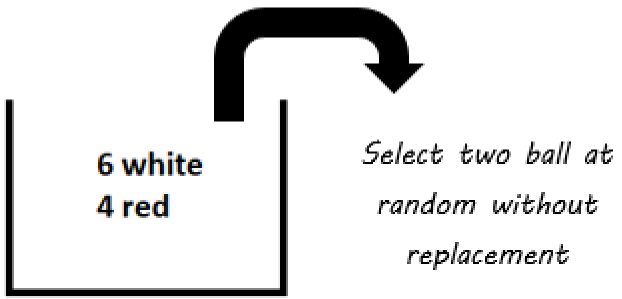
9) $P(X \geq 3 \mid X \leq 4) =$

10) $P(X = 7) =$

11) $E(x) =$

12) σ^2 =

Example (38):



Define the random variable x to be the number of white ball. Find probability distribution of x.

Exercise (10): Construct a probability distribution for a family with 4 children. Let X be the number of girls.

Exercise (11): The discrete random variable X has probability distribution given by the formula

$$P(X = x) = c(3 - x); \;\; x = 0, 1, 2, 3.$$

Find the value of c, then evaluate $P(X \leq 2)$.

Exercise (12): True or False.

1) The probability of an event always lies between 0 and 1.
2) The sum of probabilities of all possible outcomes in a sample space is always greater than 1.
3) If two events are mutually exclusive, their intersection is zero.
4) An independent event means that the occurrence of one event does not affect the probability of the other event.
5) The total probability of all values in a probability distribution must sum to 1.
6) The variance of a probability distribution measures the spread of data around the mean.